ON POSITIVE-OPERATOR-VALUED MEASURE FOR PHASE MEASUREMENTS

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Abstract

The unnormalizable Susskind-Glogower (SG) phase eigenstates, which generate a positive-operator-valued measure (POVM) for phase measurements, do not have the properties of definite phase states, because they are not eigenstates of the conjugate of the SG phase operator. The probability distribution associated by the SG-POVM with a quantum state can be interpreted as a phase distribution in the SG formalism only if the state has less overlap with the vacuum state. While alleged problems with the SG operator are justified by a discussion of the effect proposed by Nieto [Phys. Lett. A **60**, 401 (1977)], due to measuring a phase difference via the media of reference phases, a well-behaved phase operator with the conventional nonunitary approach can be constructed by introducing the exact form of the amplitude term in the decomposition of annihilation operator of the harmonic oscillator.

The positive-operator-valued measure (POVM), or probability operator measure (POM), generated by the unnormalizable Susskind-Glogower (SG) phase eigenstates, is a useful theoretical tool for investigating quantum phase measurements. [1] Lane *et al.* [2] have recently performed interesting computer simulations of maximum-likelihood scheme of multiple phase measurements, which utilizes the SG-POVM, and the results suggest a limiting phase sensitivity less than $1/N$, where N is the average number of detected photons. In the introduction of their paper they have also made statements on the properties of the SG-POVM. We feel it would be useful to make some clarification on the concept of the SG-POVM.

In their introduction they begin with a discussion of the SG operator and its alleged problems [3] from the fact that the SG operator

$$\begin{aligned} \widehat{e^{i\phi}} &= \widehat{\cos\phi} + i\ \widehat{\sin\phi} \\ &\equiv \sum_{n=0}^{\infty} |n\rangle\langle n+1| = (\hat{N}+1)^{-1/2}\,\hat{a}, \end{aligned} \tag{1}$$

has (unnormalizable) eigenstates

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle , \tag{2}$$

where

$$\widehat{e^{i\phi}}|e^{i\phi}\rangle = e^{i\phi}|e^{i\phi}\rangle ; \tag{3}$$

these provide a resolution of the identity operator

$$\int_{\phi_0}^{\phi_0+2\pi} \frac{d\phi}{2\pi}|e^{i\phi}\rangle\langle e^{i\phi}| = \hat{1}. \tag{4}$$

Because the eigenstates Eq. (2) have definite phase as suggested in Ref. [4], the completeness property Eq. (4) is ready to associate a *phase probability distribution* with any quantum state $|\psi\rangle$,

$$P_{|\psi\rangle}(\phi) \equiv \langle\psi|\left(\frac{1}{2\pi}|e^{i\phi}\rangle\langle e^{i\phi}|\right)|\psi\rangle = \frac{1}{2\pi}|\langle e^{i\phi}|\psi\rangle|^2 . \tag{5}$$

Since

$$\int_{\phi_0}^{\phi_0+2\pi} d\phi \cos^2\phi\, P_{|\psi\rangle}(\phi) = \langle\psi|(\widehat{\cos\phi})^2|\psi\rangle + \frac{1}{4}|\langle 0|\psi\rangle|^2 , \tag{6}$$

(an identical relation holds also for $\sin\phi$), they raised a puzzle for argument that the vacuum-state's second moments of $\cos\phi$ and $\sin\phi$, calculated with respect to $P_{|0\rangle}(\phi)$, equal 1/2, whereas the ideal measurements of that for the Hermitian operators $\hat{\cos\phi}$ and $\hat{\sin\phi}$ equal 1/4.

However, contrary to what they believed, the eigenstates Eq. (2) in fact do not have the properties of states with definite phase. This can be easily seen from the fact that the conjugate of $\hat{e^{i\phi}}$

$$(\hat{e^{i\phi}})^\dagger = \sum_{n=0}^{\infty} |n+1><n| \tag{7}$$

does not have the states Eq. (2) as eigenstates:[5]

$$(\hat{e^{i\phi}})^\dagger |e^{i\phi}> = e^{-i\phi} [|e^{i\phi}> - |0>]. \tag{8}$$

Consequently, in a strict sense, Eq. (5) cannot be interpreted as a phase probability distribution,[6] but rather, a probability distribution of finding a complex-valued $e^{i\phi}$ corresponding to the state $|e^{i\phi}>$ in a given state $|\psi>$ in the theory of generalized measurements, as is exactly stated in Eq. (5) itself.

Nevertheless, in the representation formed by the SG eigenstates (2), when a state $|\psi>$ has less overlap with the vacuum state $|0>$, from Eq. (8) it can be seen that the matrix element

$$<\psi|(\hat{e^{i\phi}})^\dagger |e^{i\phi}> = e^{-i\phi} [<\psi|e^{i\phi}> - <\psi|0>] \approx e^{-i\phi} <\psi|e^{i\phi}> . \tag{9}$$

From this condition it is then possible to relate $P_{|\psi>}(\phi)$ approximately as a phase probability distribution to the state $|\psi>$ to describe the statistics of an ideal phase measurement. This is in agreement with the viewpoint in a recent paper on phase states and operators [7] that a quantum state $|\psi>$ can be assigned a definite phase "only if it is a *large-amplitude localized* state."

In this way, we can suggest that the SG-POVM is a convenient way of packing the information of phase variables with computational advantages, from which one derives the probability distribution for an ideal phase measurement *whenever* the concerned quantum state $|\psi>$ is far from the vacuum state. From the probability distribution one can calculate the expectation value of any function of ϕ, for instance, as shown in Eq.(6), *as long as* the overlap in the second term of the right-hand side tends to zero. The physical nature of phase variables reflected from the SG-POVM, however, is the same as that in terms of the SG operator as extensively reviewed by Carruthers and Nieto,[8] since in quantum mechanics a representation is used only to minimize the labor in which the representatives of the more important vectors and operators occurring in the problem are as simple as possible.

Thus, the alleged problems with the SG phase operator remain unsolved that the vacuum-state second moments of $\hat{\cos}\phi$ and $\hat{\sin}\phi$ equal 1/4, although they should not come from expectations, but with physical justification.[2] Indeed, the number states are states with random phase might merely be one's expectation with the "naive" Dirac theory as a basis, number and phase are conjugate observables;[9] but a correct definition of the phase operator should not lead to unreasonable predictions. We shall begin the justification with physical reasoning based on arguments concerning uncertainties in phase measurements. Nieto has successfully analyzed the reported phase measurements [10] in terms of phase-difference operators:[11]

$$\hat{\cos}_{12}\phi = \hat{\cos}_1\phi\,\hat{\cos}_2\phi + \hat{\sin}_2\phi\,\hat{\sin}_1\phi\,, \tag{10}$$

$$\hat{\sin}_{12}\phi = \hat{\sin}_1\phi\,\hat{\cos}_2\phi - \hat{\sin}_2\phi\,\hat{\cos}_1\phi\,. \tag{11}$$

Because an absolute phase is not a physically measurable quantity, a phase measurement is usually performed by measuring the phase difference relative to a reference field. Assuming that the radiation fields are in coherent states

$$|\alpha_i\rangle = \exp[-\frac{1}{2}|\alpha_i|^2]\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|n\rangle\,, \tag{12}$$

where α_i is complex and $\langle\alpha_i|\hat{N}|\alpha_i\rangle = |\alpha_i|^2$ is the average photon number, one gets the phase-difference uncertainty of two uncorrelated fields 1 and 2

$$\begin{aligned}(\Delta\phi)^2_{\mathrm{PD}} &= (\Delta\hat{\cos}_{12})^2 + (\Delta\hat{\sin}_{12})^2\\ &= 1 - \frac{1}{2}e^{-N_1} - \frac{1}{2}e^{-N_2} - \psi(N_1)\psi(N_2)\,,\end{aligned} \tag{13}$$

$$\psi(N_i) = N_i e^{-2N_i}\Big[\sum_{n=0}^{\infty}\frac{N_i^{\,n}}{n!(n+1)^{1/2}}\Big]^2\,,\quad (i = 1, 2). \tag{14}$$

Thus, the phase uncertainty [cf. $(\Delta\phi)^2_{\mathrm{P}}$] of a radiation field, for example, field 1, can be inferred by measuring the phase difference relative to a reference field 2 with a well-defined classical phase as $N_2 \to \infty$. In a real case, the average photon number N_2 is always finite and the phase uncertainty of the reference field also contributes, although one should not expect in the nonunitary approach that $(\Delta\phi)^2_{\mathrm{PD}}$ is a simple sum of phase uncertainties of fields 1 and 2 as in the Hermitian phase approach.[12, 13] However, this is not the case as shown in Fig. 1, in which the SG formalism predicts that in the small-photon-number region $(\Delta\phi)^2_{PD}$ approaches to zero, even though both the phase uncertainties of uncorrelated fields 1 and 2 tend to their maximum value 1/2.

Generally, one can consider an effect, also discussed by Nieto,[11] due to measuring a phase difference via the media of reference phases. The phase operators $\hat{COS}_{1k}$ and $\hat{SIN}_{1k}$, by going through the steps 2, 3, ..., k-1, are given as

$$\hat{COS}_{1k} = \hat{\cos}_{1k}\phi \prod_{j=2}^{k-1} [(\hat{\cos}_j\phi)^2 + (\hat{\sin}_j\phi)^2] = \hat{\cos}_{1k}\phi \prod_{j=2}^{k-1} [1 - \frac{1}{2} P_0(0)], \qquad (15)$$

$$\hat{SIN}_{1k} = \hat{\sin}_{1k}\phi \prod_{j=2}^{k-1} [(\hat{\cos}_j\phi)^2 + (\hat{\sin}_j\phi)^2] = \hat{\sin}_{1k}\phi \prod_{j=2}^{k-1} [1 - \frac{1}{2} P_0(0)], \qquad (16)$$

where $P_0(0)$ is the projector onto the vacuum state. In Fig. 2 the uncertainties of phase difference $(\Delta\phi)^2_{PD\,k}$ via different steps are calculated in radiation fields described by identical coherent states (12)

$$\begin{aligned}(\Delta\phi)^2_{PD\,k} &= (\Delta\hat{COS}_{1k})^2 + (\Delta\hat{SIN}_{1k})2 \\ &= [1 - e^{-N}] \prod_{j=2}^{k-1} [1 - \frac{3}{4} e^{-N}] - [\psi(N)]^2 \prod_{j=2}^{k-1} [1 - \frac{1}{2} e^{-N}]^2. \end{aligned} \qquad (17)$$

From Fig. 2 it can be recognized that the extremely small $(\Delta\phi)^2_{PD\,k}$ range stretches into the large-N region with increasing steps.

The seriousness of the problem becomes more explicit when studying a case in which vacuum fields are chosen in the intermediate steps. The whole curve of $(\Delta\phi)^2_{PD\,k}$ will be quickly lowered to the zero level with only a few steps (by a factor $1/4^{k-2}$, k-2 is the intermediate step number), because the expectation of each term added to the multiplication in the right-hand sides of Eqs. (15) and (16) is always less than unity (especially, 1/2 for the vacuum). This prediction of reduced uncertainty with increasing reference steps turns out contrary to measurement theories (both quantum and classical) that the uncertainty is accumulated through various participating steps in a measurement process.

Thus, this *Nieto effect* suggests a criterion to test a phase operator whether with proper physical implication, i.e., the expectation value of the sum of the second moments of the Hermitian real and imaginary parts in any quantum state $|\psi\rangle$ must be no less than unity.

To get a reasonable definition of the phase operator with the *conventional* nonunitary approach, let us return to the origin of the phase problem from which the SG operator came,[14] i.e., Dirac's decomposition of the annihilation (or creation) operator corresponding to the classical procedure of separating the complex amplitude into a real amplitude and a phase factor. Generally, because of the particular number spectrum N of the harmonic oscillator that runs only from 0 to ∞, a physical decomposition depends on cautious selection, around the lower bound of $N = 0$, of not only the order of separation but also the form of the amplitude. For example, when the form of amplitude proportional to $\hat{N}^{1/2}$ is chosen in the Dirac formalism, the action of the Dirac phase factor

$$\hat{e_D^{i\phi}} = \hat{a}\hat{N}^{-1/2} \tag{18}$$

on the vacuum is indeterminate due to N including the zero eigenvalue and therefore no inverse. This difficulty was removed by Susskind and Glogower when replacing the definition Eq. (18) with the unambiguous form Eq. (1).

However, by careful examination of the Hamiltonian structure $\hat{H}$ of the harmonic oscillator one finds that the form of the amplitude should be proportional to $(\hat{N} + 1/2)^{1/2}$, where the factor 1/2 corresponds to the zero-point energy of the vacuum. In fact, the *exact* amplitude operator of the oscillator (unit mass) takes the form

$$\hat{A} \equiv (2\hat{H})^{1/2}/\omega = A_0\,(\hat{N} + 1/2)^{1/2} \ , \tag{19}$$

where $A_0 = (2\hbar/\omega)^{1/2}$.

Consequently, the decomposition with the amplitude proportional to $(\hat{N} + 1/2)^{1/2}$ yields[15]

$$\hat{E} = (\hat{N} + 1/2)^{1/2}\,\hat{a} \ , \tag{20}$$

with the following properties

$$\hat{E}\hat{E}^{\dagger} = 1 + \frac{1}{2\hat{N} + 1} \ , \tag{21}$$

$$\hat{E}^{\dagger}\hat{E} = 1 + \frac{1}{2\hat{N} - 1} \ , \tag{22}$$

and

$$[\hat{C}, \hat{S}] = \frac{1}{2i}\left(\frac{1}{2\hat{N} + 1} - \frac{1}{2\hat{N} - 1}\right) \ , \tag{23}$$

$$\hat{C}^2 + \hat{S}^2 = 1 + \frac{1}{2}\left(\frac{1}{2\hat{N} + 1} + \frac{1}{2\hat{N} - 1}\right) , \tag{24}$$

where $\hat{C}$ and $\hat{S}$ are the Hermitian real and imaginary parts of $\hat{E}$. From Eq. (24) it is seen that the definition Eq. (20) meets the above Nieto criterion.[16]

The fact that the exact amplitude is proportional to $(\hat{N} + 1/2)^{1/2}$ therefore explains the problems in defining a phase operator. The Dirac decomposition with $\hat{N}^{1/2}$ is equivalent to taking the vacuum energy as zero. For an empty field the phase will be no longer meaningful, and the decomposition (18) becomes unperformable with the vacuum. The SG decomposition with $(\hat{N}+1)^{1/2}$ corresponds to taking the zero-point energy as $\hbar\omega$, which overcounts the actual one by $\frac{1}{2}\hbar\omega$. As a result, abnormal smaller phase uncertainty in a given quantum state $|\psi>$ develops. On the other hand, The factor 1/2 in the amplitude Eq. (19) prevents the phase definition from ambiguity when applied to the vacuum. Eqs. (21) and (22) show the non-unitary property of $\hat{E}$ in any state of the oscillator, which approaches unity gradually with increasing N as required by the correspondence principle. This nonunitarity leads to the relations (23) and (24), indicating an extra uncertainty added to the phase uncertainty that can be expected from the simpler Dirac theory, and guarantying the satisfaction of the Nieto criterion. As a matter of fact, the non-commuting property of the phase operator is intrinsic within the framework of quantum mechanics as a direct result of decomposition of the *non-commuting* annihilation operator, which has been recently re-discussed from a quantum phase anomaly viewpoint.[17, 18]

Thus, we can also conclude that, with the further investigations of quantum phase measurements, one no longer rests content with correspondence-principle-type arguments in the large-N limit, such as imposed by the Lerner criterion,[19] but more lays stress on the small-N behavior of phase operators.[13] With the exact amplitude form in the decomposition of the annihilation operator, which has rightly taken the zero-point-energy effect into account, a nonunitary phase operator free from alleged problems can be constructed.

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15. It was realized in many phase definitions that the zero-point energy must be taken into consideration, for example, in the Lerner-Lynch formalism [E. C. Lerner, Nuovo Cimento **56B**, 183 (1968) ; **57B**, 251 (1968); R. Lynch, J. Opt. Soc. Am. **B3**, 1006 (1986)]. However, the definition corresponds to a decomposition of $\hat{a}$ with a rather complicated amplitude form proportional to $2/[(\hat{N} + 1/2)^{-1/2} + (\hat{N} + 3/2)^{-1/2}]$. Moreover, it is easy to find that the definition fails to satisfy the above Nieto criterion.

16. One must also consider the separation order in the decomposition. Another possible definition of $\hat{E}$ might be obtained by reversing the order of the product in the right-hand side of Eq. (20), which, however, fails the Nieto criterion.

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Figure captions

Fig. 1. The phase-difference uncertainty $(\Delta\phi)^2{}_{\mathrm{PD}}$ of a radiation field in a coherent state $|\alpha_1>$ relative to a reference field in $|\alpha_2>$. $(\Delta\phi)^2{}_{\mathrm{PD}}$ tends to the phase uncertainty of field 1 as $N_2 \to \infty$.

Fig. 2. Phase-difference uncertainties $(\Delta\phi)^2{}_{\mathrm{PD}k}$ via different intermediate steps of radiation fields in the identical coherent states (solid curves): a) without intermediate step; b) 10 steps; c) 10^2 steps; d) 10^3 steps; e) 10^4 steps. The dashed curve shows the phase uncertainty of a single field. In the large-N limit, $(\Delta\phi)^2{}_{\mathrm{PD}\,k}$ tend to the double of the phase uncertainty of a single field. If vacuum fields in stead of coherent-state fields are used in the intermediate steps, the curve (a) in the whole range of N will be rapidly lowered to the zero level with only a few steps.

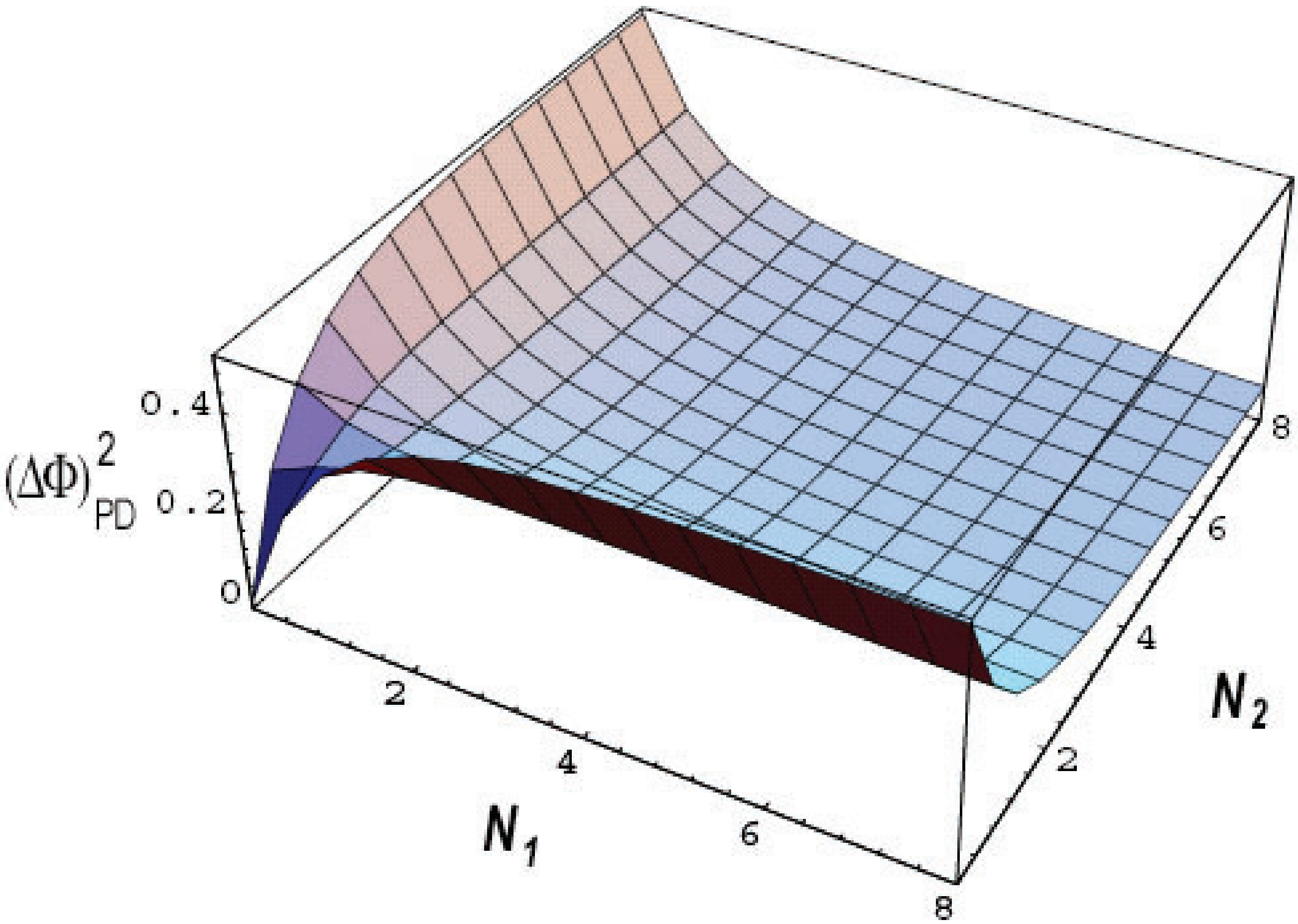

Fig.1 Zheng and Kobayashi

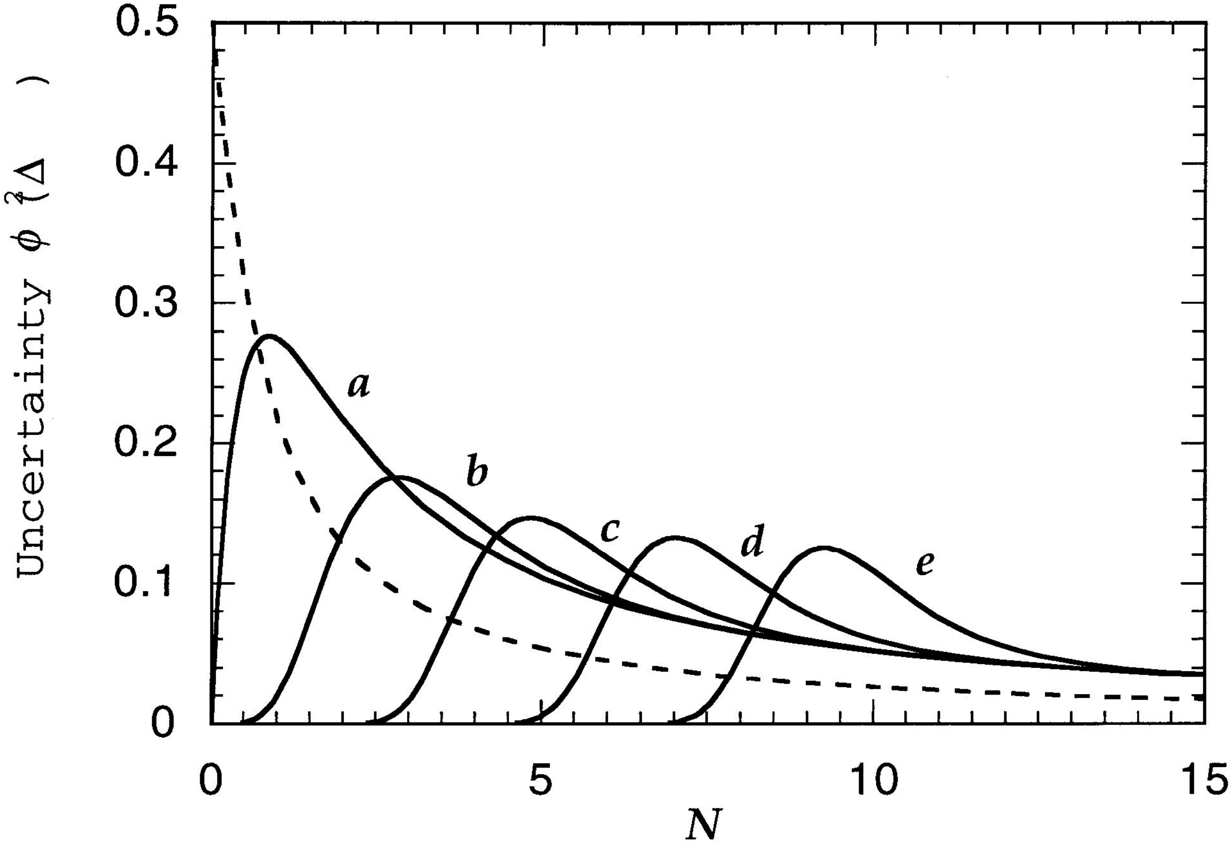

Fig. 2
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